

Separation of variables for local symmetrical flows

Lang Xia

langshah@ufl.edu, Department of Mechanics, Fudan University, Shanghai 200433, China

Abstract

Separation of variables is effective method for solving ordinary and partial differential equations. We examine some topological manifolds in flows and get a conclusion that it can be applied in separating variables of differential equations. Then we give an example of simplifying Cauchy momentum equation.

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1 Introduction

Normally, We call a differential equation

$$\frac{dx}{dt} = f(x, t)$$

has separable variables[1], if $f(x, t) = g(x)h(t)$ for some functions g and h , where g depends only on x and h depends only on t .

This method, named Separation of variables, is effective for solving ordinary and partial differential equations^[2,3,4]. However, it's limited in solving linear equations, at the same time, it requires homogeneous boundary conditions which can be rarely satisfied in the most of time. Traditionally, people make the effort to transfer the equations into or similar to linear equations and the conditions into homogeneous boundary conditions. These are tough problems when the equations and conditions are very complex. In stead of doing so, we propose to analyze the physical topological characteristics of flows according to V.I.Arnold^[5]. Basing on such ideas, Lie group is introduced in treating flows

directly. And we obtain a conclusion of some flows, where the methods of separation of variables can be used. As Cauchy momentum equation is a general form of flows, we set it as an example for illustrating ours conclusion. Also, topological characteristics of flows are briefly discussed.

2 Proof and Result

We call local symmetrical flows in this paper means that motions form some kinds of Lie groups at a time, such as rings($S^1(1)$), cylinder($\mathbb{R} \times S^1(1)$) and torus($\mathbb{T}^2 = S^1 \times S^1$) e.t.c.

Theorem. *For any $x_0 \in \Omega$ on the local symmetrical flows at time t_0 , there exists $G(t)$, $H(x)$ such that $y = G(t)H(x)x_0$. Here $|t - t_0| \leq \delta$, $\lim \delta = 0$, y is an arbitrary point in Ω at time t , and $G(t)$, $H(x)$ are Lie groups respectively.*

Remark. y is not necessary on the local symmetrical flows.

Proof. We know that a fibre bundle $\xi = (E, P, B)$ is a vector bundle if $E \cong B \times \mathbb{R}^n$ locally^[6], now let $B = \mathbb{R}$, giving a point $t_0 \in W_0 \subset B$, W_0 is an open neighbourhood, t is also a point in W_0 other than t_0 . Let

$$F = P^{-1}(t) = \mathbb{R}^3, \quad P^{-1}(W_0) \cong W_0 \times F$$

If $U_0 \subset F_0 = P^{-1}(t_0)$ is an open neighbourhood of a certain identity x_0 (we assume that some kinds of symmetrical flows in F_0), and X_i are tangent vectors on U_0 . Define

$$[X_i, X_j] = X_i \bullet X_j, \quad (i, j = 1, 2, 3)$$

where “ \bullet ” stands for any kind of multiplications, such that

$$L = \{X_1, X_2, X_3\}$$

is some Lie algebra, and so there is a local Lie group $H(x)$ ^[7], we suppose it is isomorphism to U , which

$$x_0 \in U \subset U_0, \quad \phi : H \times U \longrightarrow U, \quad \phi(H, x) = Hx$$

and we have

$$x = H(x)x_0$$

where $x \in U$.

Then, for another point $t \in W_0$, as \mathbb{R} is a nature abelian Lie group, we have

$$t = g(t)t_0$$

In terms of the continuity hypothesis, there exists an open subset $V_0 \subset P^{-1}(t)$ which is homomorphic to U_0 . write as

$$V_0 = \varphi_t U_0$$

where φ_t is one-parameter group.

There also exists $V \subset V_0$ that is homomorphic to U , in other words

$$y|_V = \varphi_t x|_U$$

Since ξ is local trivial, we can choose V such that V is isomorphic to U .

On the other hand

$$P^{-1}(t) = P^{-1}(g(t)t_0)$$

and it is easy to prove

$$P^{-1}|_V(g(t)t_0) = \epsilon P^{-1}|_\epsilon(g(t))P^{-1}|_U(t_0)$$

where ϵ is a constant. Obviously,

$$y|_V = G(t)H(x)x_0 \quad \text{as } G(t) = \epsilon P^{-1}|_\epsilon(g(t))$$

Consider every point closes to V in the fluid, Let

$$K = \{x \mid d(x, V) < \delta, \lim \delta = 0\}$$

Since the properties of points in K are very similar to points in V , so we can expand the domain to $N = K + V$ such that

$$P^{-1}|_N(t) = \epsilon P^{-1}|_\epsilon(g(t))P^{-1}|_U(t_0)$$

$$\forall y \in K, y = G(t)H(x)x_0$$

□

Ours above proof infer a kind of symmetry when applying to fluid. The Lie algebra L , which represents velocity field of fluid particle on the local symmetrical flows, is well defined as the product gives out some information of fluid such as rotations and deformations(when “ \bullet ” is cross product “ \times ”). It is not difficult to see that the conclusion “splits” time off space on the local symmetrical flows. Where $G(t)$ controls the changing of time and $H(x)$ is in charge of space deformation.

3 Discussion

Obviously, using Lie groups with a kind of multiplications, we simplified the partial differential equation into two ordinary differential equations(here we mean that t, \mathbf{x} are independent) without requiring linear equation or homogeneous boundary conditions. Because $\mathbf{v}(\mathbf{x}, t)$ can be written as $\mathbf{v}(\mathbf{x}, t) = \mathbb{G}'\mathbb{H}\mathbf{x}^0$ naturally near the local symmetrical flows.

And if in the fluid where there is a time span

$$|t - t_0| \leq \delta, \lim \delta = 0$$

such that

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t_0)$$

From [5] we know that some kinds of topological manifolds can be classified, such as rings($S^1(1)$), cylinder($\mathbb{R} \times S^1(1)$) and torus($\mathbb{T}^2 = S^1 \times S^1$) e.t.c. Generally, one can always find some differentiable manifolds, and when these manifolds are Lie groups(or diffeomorphic to lie groups), then $H(x)$ can be identified easily.

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